6. Derivatives Part 2

In this lecture, we will discuss

- Linear Approximation
    - Review of linear approximation of $f : \mathbb{R} \to \mathbb{R}$
    - Linear approximation of $f : \mathbb{R}^2 \to \mathbb{R}$
        - Formula $L_{(a,b)}(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)$
        - Geometric Interpretation: Tangent Plane
    - Linear Approximation of $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$
- Properties of Derivatives
    - Basic Properties of Derivatives
    - Chain Rule

Linear Approximation

Review: Linear Approximation of $f : \mathbb{R} \to \mathbb{R}$

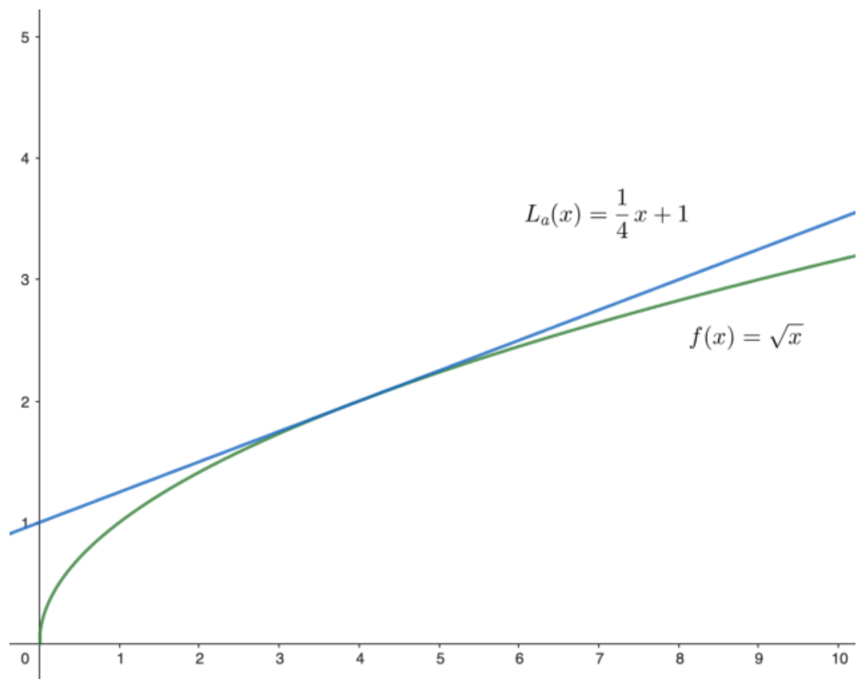
Let $f : \mathbb{R} \to \mathbb{R}$ be a differetiable function at a point a. Recall the *linear appproximation* or the *linearization* of f at a:

$$L_a(x) = f(a) + f'(a)(x - a) \tag{1}$$

Geometrically, L_a represents the equation of the line tangent to the graph of f at a. It is written in point-slope form: the point is $(a, f(a))$, and $f'(a)$ is the slope.

For example, let $f(x) = \sqrt{x}$ and $a = 4$.

- $f(4) = \sqrt{4} = 2$. $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ thus $f'(4) = \frac{1}{4}$.
- So $L_4(x) = 2 + \frac{1}{4}(x - 4) = \frac{1}{4}x + 1$.
- Using this, $\sqrt{4.04} \approx L(4.04) = (1/4)(4.04) + 1 = 2.01$



- $L_a(x)$ is a good approximation to $f(x)$ near a, i.e., the tangent line is a good approximation to the curve $y = f(x)$ near a.
- Recall it is a special case for the Taylor's thorem, which states

  $$f(x) = f(a) + f'(a)(x - a) + R_2$$

  where R_2 is the remainder term.

Linear Approximation of $f : \mathbb{R}^2 \to \mathbb{R}$

- **1. Deriving the formula**

  Next, let's take a function $f : \mathbb{R}^2 \to \mathbb{R}$ differentiable at point $\mathbf{a}$, with the notation $\mathbf{x} = (x, y)$ and $\mathbf{a} = (a, b)$, we generalize the term $f'(a)(x-a)$ in Eq (1) to

Recall from previous Lecture

$$Df(\vec{a}) = \left[\frac{\partial f}{\partial x}(a,b) ,\ \frac{\partial f}{\partial y}(a,b)\right]$$

$$\vec{x} - \vec{a} = \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$\begin{aligned} Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) &= \begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \end{bmatrix} \cdot \begin{bmatrix} x-a \\ y-b \end{bmatrix} \\ &= \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) \end{aligned}$$

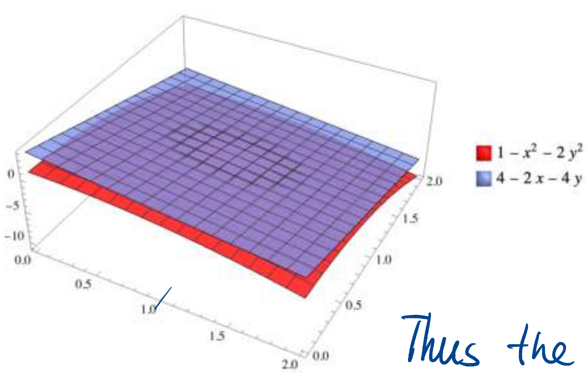
Also, the generalized form of Eq (1) , $L_a(x) = f(a) + f'(a)(x-a)$, is

$$L_{\mathbf{a}}(x,y) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a})$$

Therefore, we have

$$\boxed{L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)} \qquad (2)$$

Example 1 Let $f(x,y) = 1 - x^2 - 2y^2$ and $\mathbf{a} = (1,1)$. Find the linear approxmation of f near $\mathbf{a}$.



ANS: We compute

$$\frac{\partial f}{\partial x} = -2x, \quad \text{then} \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = -2$$

$$\frac{\partial f}{\partial y} = -4y, \quad \text{then} \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = -4.$$

Thus the linear approximation of f near $\vec{a}$ is

$$L_{(1,1)}(x,y) = f(\cancel{1,1})^{-2} - 2\cdot(x-1) - 4(y-1)$$

$$\Rightarrow L_{(1,1)}(x,y) = 4 - 2x - 4y$$

From the above graph, we can see the plane L is a good approximation to the graph of f near the point $(1,1)$.

$L_{(1,1)}(x,y)$ is called the tangent plane of f at $(1,1)$.

(Check the next page for the def of tangent plane)

You can try to plot the image in **Example 1** using Mathematica by typing the following code.

```
Plot3D[{1 - x^2 - 2 y^2, 4 - 2 x - 4 y}, {x, 0, 2}, {y, 0, 2},
 PlotTheme -> "Scientific", PlotLegends -> "Expressions",
 PlotStyle -> {Directive[Opacity[0.8], RGBColor[1, 0, 0]],
   Directive[Opacity[0.8], blue]}]
```

- **2. Geometric Interpretation: Tangent Plane**
  - Geometrically, linear approximation represents the equation of a plane in $\mathbb{R}^3$ (e.g., $z = 4 - 2x - 4y$ in **Example 1**).
  - This plane has the point $(a, b, f(a, b)) = \big(a, b, L_{(a,b)}(a, b)\big)$ in common with the graph of f, which is called a *tangent plane*.
  - It is defined by the Eq (2): $z = L_{(a,b)}(x, y) = f(a, b) + \dfrac{\partial f}{\partial x}(a, b) \cdot (x - a) + \dfrac{\partial f}{\partial y}(a, b) \cdot (y - b)$.

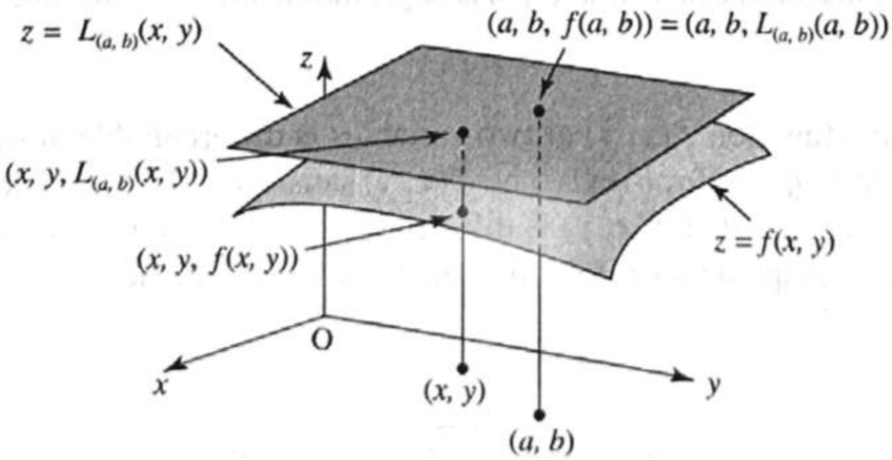


Figure 2.49 Linear approximation $L_{(a,b)}(x, y)$ of a function $f(x, y)$ at (a, b).

Linear Approximation of $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$

In general, let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at $\mathbf{a}$.

Define the *linear approximation* $L_{\mathbf{a}}(\mathbf{x})$ of $\mathbf{F}(\mathbf{x})$ at $\mathbf{a}$ or the *linearization* of $\mathbf{F}(\mathbf{x})$ at $\mathbf{a}$

$$L_{\mathbf{a}}(\mathbf{x}) = \mathbf{F}(\mathbf{a}) + D\mathbf{F}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

It is a good approximation of $\mathbf{F}$ near $\mathbf{a}$.

Below are two exercises similar to some of our WebWork homework.

Exerecise 2. (Related to WebWork Q2)

Find the equation of the tangent plane to

$$z = e^x + y + y^3 + 10$$

at the point $(0, 2, 21)$.

Answer. We have

$$z = f(x, y) = e^x + y + y^3 + 10.$$

The partial derivatives are

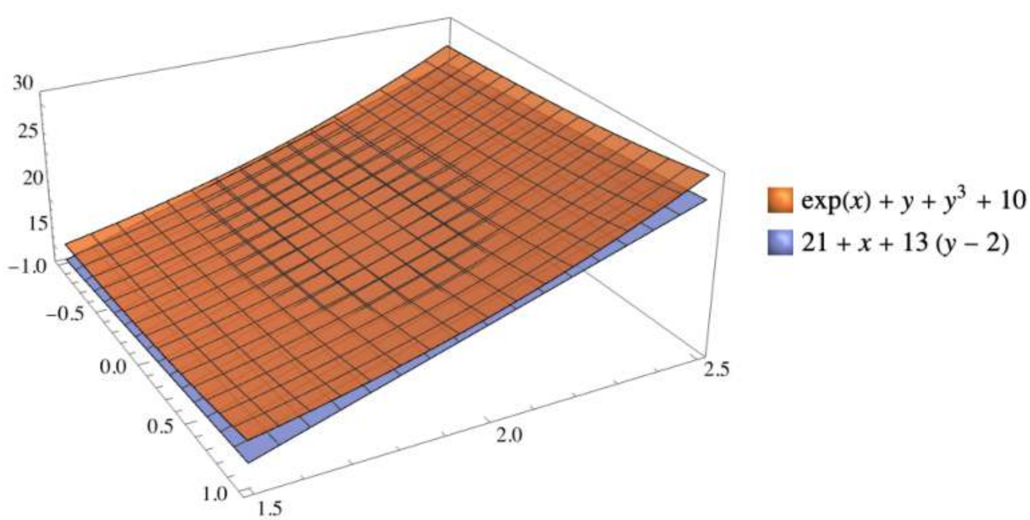
$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,2)} = \left.e^x\right|_{(x,y)=(0,2)} = 1,$$

and

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,2)} = \left.1 + 3y^2\right|_{(x,y)=(0,2)} = 13.$$

So the equation of the tangent plane is

$$L_{(a,b)}(x, y) = f(0, 2) + \frac{\partial f}{\partial x}(0, 2) \cdot (x - 0) + \frac{\partial f}{\partial y}(0, 2) \cdot (y - 2)$$
$$\implies z = 21 + 1(x - 0) + 13(y - 2) = 21 + x + 13(y - 2).$$



Exercise 3. (Related to WebWork Q4)

Find the linearization of the function $f(x,y) = \sqrt{24 - 3x^2 - 3y^2}$ at the point $(-1, 2)$. Then use the linear approximation to estimate the value of $f(-1.1, 2.1)$.

Answer.

We have $f(x,y) = \sqrt{24 - 3x^2 - 3y^2} = (24 - 3x^2 - 3y^2)^{1/2}$

The partial derivatives are

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(-1,2)} = -\left.\frac{3x}{\sqrt{-3x^2 - 3y^2 + 24}}\right|_{(x,y)=(-1,2)} = 1,$$

and

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(-1,2)} = -\left.\frac{3y}{\sqrt{-3x^2 - 3y^2 + 24}}\right|_{(x,y)=(-1,2)} = -2.$$

So the equation of the tangent plane is

$$\begin{gathered} L_{(a,b)}(x,y) = f(-1,2) + \frac{\partial f}{\partial x}(-1,2) \cdot (x - (-1)) + \frac{\partial f}{\partial y}(-1,2) \cdot (y - 2) \\ \implies z = 3 + 1(x+1) - 2(y-2) = 8 + x - 2y. \end{gathered}$$

Thus

$$f(-1.1, 2.1) \approx 8 + (-1.1) - 2(2.1) = 2.7.$$

Properties of Derivatives

Theorem 1. Properties of Derivatives

(a) Assume that the functions $\mathbf{F}, \mathbf{G} : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ are differentiable at a $\in U$. Then the sum $\mathbf{F} + \mathbf{G}$ and the difference $\mathbf{F} - \mathbf{G}$ are differentiable at $\mathbf{a}$ and

$$D(\mathbf{F} \pm \mathbf{G})(\mathbf{a}) = D\mathbf{F}(\mathbf{a}) \pm D\mathbf{G}(\mathbf{a})$$

(b) If the function $\mathbf{F} : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{a} \in U$ and $c \in \mathbb{R}$ is a constant, then the product $c\mathbf{F}$ is differentiable at $\mathbf{a}$ and

$$D(c\mathbf{F})(\mathbf{a}) = cD\mathbf{F}(\mathbf{a}).$$

(c) If the real-valued functions $f, g : U \subseteq \mathbb{R}^m \to \mathbb{R}$ are differentiable at $\mathbf{a} \in U$, then their product fg is differentiable at $\mathbf{a}$ and

$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}).$$

(d) If the real-valued functions $f, g : U \subseteq \mathbb{R}^m \to \mathbb{R}$ are differentiable at $\mathbf{a} \in U$, and $g(a) \neq 0$, then their quotient f/g is differentiable at a and

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}.$$

(e) If the vector-valued functions $\mathbf{v}, \mathbf{w} : U \subseteq \mathbb{R} \to \mathbb{R}^n$ are differentiable at $a \in U$, then their dot (scalar) product $\mathbf{v} \cdot \mathbf{w}$ is differentiable at a and

$$(\mathbf{v} \cdot \mathbf{w})'(a) = \mathbf{v}'(a) \cdot \mathbf{w}(a) + \mathbf{v}(a) \cdot \mathbf{w}'(a).$$

(f) If the vector-valued functions $\mathbf{v}, \mathbf{w} : U \subseteq \mathbb{R} \to \mathbb{R}^3$ are differentiable at $a \in U$, their cross (vector) product $\mathbf{v} \times \mathbf{w}$ is differentiable at a and

$$(\mathbf{v} \times \mathbf{w})'(a) = \mathbf{v}'(a) \times \mathbf{w}(a) + \mathbf{v}(a) \times \mathbf{w}'(a)$$

Example 4.

Let $f(x,y,z) = xy + e^z$ and $g(x,y,z) = y^2 \sin z$. Use the product rule to compute $D(fg)(0,1,\pi)$.

ANS: By the Product Rule (Th1 (c)), we have

$$\circledast\ D(fg)(0,1,\pi) = g(0,1,\pi)\, Df(0,1,\pi) + f(0,1,\pi)\, Dg(0,1,\pi)$$

Recall $Df(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} y & x & e^z \end{bmatrix}$

$Dg(x,y,z) = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 2y\sin z & y^2\cos z \end{bmatrix}$

Thus $\circledast \Rightarrow$

$$D(fg)(0,1,\pi) = 1^2 \sin\pi \begin{bmatrix} 1 & 0 & e^\pi \end{bmatrix} + e^\pi [0, 0, -1]$$
$$= [0, 0, -e^\pi]$$

Review of the Chain Rule for functions $\mathbb{R} \to \mathbb{R}$

Recall in calculus, we have the Chain Rule

$$\boxed{(f \circ g)'(x) = f'(g(x))g'(x)} \tag{3}$$

$[\sin(3x^2+x)]'$
$= (\cos(3x^2+x)) \cdot (6x+1)$

Other notation: If $y = f(u)$ and $u = g(x)$, that is, $y = f(g(x))$. We sometimes write the chain rule in the following form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{4}$$

The generalized chain rule is summarized in the following theorem.

Theorem 2 Chain Rule

Suppose that $\mathbf{F} : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{a} \in U$, U is open in $\mathbb{R}^m$, $\mathbf{G}; V \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $\mathbf{F}(\mathbf{a}) \in V$, V is open in $\mathbb{R}^n$, and $\mathbf{F}(U) \subseteq V$ (so that the composition $\mathbf{G} \circ \mathbf{F}$ is defined). Then $\mathbf{G} \circ \mathbf{F}$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{G} \circ \mathbf{F})(\mathbf{a}) = D\mathbf{G}(\mathbf{F}(\mathbf{a})) \cdot D\mathbf{F}(\mathbf{a}),$$

where $\cdot$ denotes matrix multiplication.

Example 5. (Related to WebWork Q5)

Let $\mathbf{r}(t) = \langle e^t, e^{2t}, 5\rangle$, and $g(t) = 2t - 3$.

Compute $\frac{d\mathbf{r}}{dt}(g(t))$.

$\vec{r} : \mathbb{R} \to \mathbb{R}^3$

$g : \mathbb{R} \to \mathbb{R}$

$\vec{r} \circ g : \mathbb{R} \to \mathbb{R}^3$

ANS: By Chain Rule, we know

$$\frac{d}{dt}\vec{r}(g(t)) = r'(g(t))\, g'(t)$$

We compute

$$r'(t) = \langle e^t, 2e^{2t}, 0\rangle$$

$$r'(g(t)) = \langle e^{2t-3}, 2e^{2\cdot(2t-3)}, 0\rangle$$

Also $g'(t) = 2$

Therefore $\frac{d}{dt}\vec{r}(g(t)) = r'(g(t))\, g'(t)$

$$= \langle 2e^{2t-3}, 4e^{4t-6}, 0\rangle$$

Note we can use the Chain Rule to compute partial derivatives for composition of functions.

Example 6. (Related to WebWork Q6)

Let $f(x,y,z) = x^4y^3 + z$, $x = s^2t^3, y = s^3t$, and $z = s^2t$.

Use the Chain Rule to compute $\dfrac{\partial f}{\partial s}$.

Answer.

The generalized formula from Eq (4) we can use for this question is

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}}$$

Let's first explain how we can use chain rule to derive the above equation:

ANS: $f: \mathbb{R}^3 \to \mathbb{R}$. Let $G: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$G(s,t) = \langle x(s,t),\ y(s,t),\ z(s,t) \rangle = \langle s^2t^3,\ s^3t,\ s^2t \rangle$$

Consider the composition of the functions f and G.

$$f(G(s,t)) : \mathbb{R}^2 \to \mathbb{R}$$

By Chain Rule, we know

$$D[f(G(s,t))] = Df(G(s,t)) \cdot DG(s,t)$$

where $Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}_{1\times 3}$

$f: \mathbb{R}^3 \to \mathbb{R}$

and $DG(s,t) = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial s} \end{bmatrix}_{3\times 2}$

Note $G: \mathbb{R}^2 \to \mathbb{R}^3$

Thus $D[f(G(s,t))] = Df \cdot DG$

Definition. Jacobian Matrix $D\mathbf{F}(\mathbf{x})$

By $D\mathbf{F}(\mathbf{x})$ we denote the $n \times m$ matrix of partial derivatives of the components of $\mathbf{F}$ evaluated at $\mathbf{x}$ (provid
that all partial derivatives exist at $\mathbf{x}$). Thus,

$$D\mathbf{F}(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \frac{\partial F_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_m}(\mathbf{x}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{x}) & \frac{\partial F_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_2}{\partial x_m}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \frac{\partial F_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_m}(\mathbf{x}) \end{bmatrix}$$

The matrix $D\mathbf{F}(\mathbf{x})$ has n rows and m columns (the number of rows is the number of component functions o
$\mathbf{F}$, and the number of columns equals the number of variables).

$$\Rightarrow \begin{bmatrix} \frac{\partial f}{\partial s} & \frac{\partial f}{\partial t} \end{bmatrix}_{1\times 2} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}_{1\times 3} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s} & \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t} \end{bmatrix}$$

Each coordinate in the above eqn must be equal, so we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

To solve the question, we compute

$$\frac{\partial x}{\partial s} = 2st^3 \qquad \frac{\partial y}{\partial s} = 3s^2t \qquad \frac{\partial z}{\partial s} = 2st$$

$$\frac{\partial f}{\partial x} = 4x^3y^3 \qquad \frac{\partial f}{\partial y} = 3x^4y^2 \qquad \frac{\partial f}{\partial z} = 1$$

Thus $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$

$$= 4x^3y^3\cdot 2st^3 + 3x^4y^2\cdot 3s^2t + 2st$$

$$= 4(s^2t^3)^3(s^3t)^3\cdot 2st^3 + 3(s^2t^3)^4\cdot(s^3t)^2\cdot 3s^2t + 2st$$

$$= 8s^{16}t^{15} + 9s^{16}t^{15} + 2st$$

$$= 17s^{16}t^{15} + 2st$$

Exercise 7. (Related to WebWork Q10, Q8, Q7)

Let $f(u,v) = \sin u \cos v$, and $u = -2x^2 + 4y$, $v = 5x - 5y$. Assume $g(x,y) = (u(x,y), v(x,y))$.

1. Compute the derivative matrix $\mathbf{D}(f \circ g)(x,y)$. (Leave your answer in terms of u, v, x, y)
2. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

ANS: 1. We follow the similar discussion as Example 6.

$$D(f\circ g)(x,y) = D[f(g(x,y))]$$

$$\Rightarrow D(f\circ g)(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} \end{bmatrix} \quad \circledast$$

We compute

$$\frac{\partial f}{\partial u} = \cos u \cos v, \quad \frac{\partial f}{\partial v} = -\sin u \sin v, \quad \frac{\partial u}{\partial x} = -4x, \quad \frac{\partial u}{\partial y} = 4$$

$$\frac{\partial v}{\partial x} = 5, \quad \frac{\partial v}{\partial y} = -5$$

Thus $$D(f\circ g)(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} -4x\cos u \cos v - 5\sin u \sin v \\ 4\cos u \cos v + 5\sin u \sin v \end{bmatrix}$$

2. From 1, we know

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = -4x\cos u\cos v - 5\sin u \sin v$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 4\cos u \cos v + 5\sin u \sin v$$

Exercise 8. (Related to WebWork Q9)

If $z = (x+y)e^y$ and $x = 6t$ and $y = 1 - t^2$, find $\dfrac{dz}{dt}$.

ANS. We us the idea in Example 6 and Exercise 7.

You can probably write down the formula to compute $\frac{dz}{dt}$ without too much trouble:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

We compute the ingredients appear on the right hand side:

$$\frac{\partial z}{\partial x} = e^y, \quad \frac{\partial z}{\partial y} = \frac{\partial (xe^y + ye^y)}{\partial y} = xe^y + e^y + ye^y$$

$$\frac{\partial x}{\partial t} = 6, \quad \frac{\partial y}{\partial t} = -2t$$

Thus

$$\frac{dz}{dt} = 6e^y + (x+y+1)e^y \cdot (-2t)$$

$$= 6e^{1-t^2} - 2t(6t+1-t^2+1)\, e^{1-t^2}$$

$$= 6e^{1-t^2} - 2t \cdot (2+6t-t^2) \cdot e^{1-t^2}$$